

# Reducing the Heterotic Supergravity on nearly-Kähler coset spaces

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## Abstract

We study the dimensional reduction of the  $\mathcal{N} = 1$ , ten-dimensional Heterotic Supergravity to four dimensions, at leading order in  $\alpha'$ , when the internal space is a nearly-Kähler manifold. Nearly-Kähler manifolds in six dimensions are all the non-symmetric coset spaces and a group manifold. Here we reduce the theory using as internal manifolds the three six-dimensional non-symmetric coset spaces, omitting the case of the group manifold in the prospect of obtaining chiral fermions when the gauge fields will be included. We determine the effective actions for these cases, which turn out to describe  $\mathcal{N} = 1$  four-dimensional supergravities of the no-scale type and we study the various possibilities concerning their vacuum.

# 1 Introduction

Supergravity theories have been studied extensively over the past thirty years. In particular, exploring the possibility that superstring theories describe the real world, the task of providing a suitable compactification which would lead to a realistic four-dimensional theory has been pursued in many diverse ways.

The early attempts to reduce such theories made extensive use of Calabi-Yau (CY) manifolds, i.e. manifolds with  $SU(3)$  holonomy [1]. A reduction and truncation procedure has been developed in refs. [2] and [3], where the internal space is not specified but the general characteristics of CY manifolds are kept. However, there exist some problems with the use of CY in the reduction procedure due to their complicated geometry. Among others their metric is not known explicitly and their Euler characteristic is too large to accommodate an acceptable number of fermion generations. Moreover, in CY compactifications the resulting low-energy field theory in four dimensions contains a number of massless chiral fields, characteristic of the internal geometry, known as moduli. These fields correspond to flat directions of the effective potential and therefore their values are left undetermined. Since these values specify the masses and couplings of the four-dimensional theory, the theory has limited predictive power.

In the context of flux compactifications the recent developments have led to the study of a wider class of internal spaces, called manifolds with  $SU(3)$ -structure, that contains CYs. The general case of  $SU(3)$ -structures is of special interest since the "local Lorentz" (structure) group  $SO(6)$  of the internal space can be reduced down to  $SU(3)$  in a way that there exists a nowhere-vanishing globally-defined spinor. In the case of CY manifolds this spinor is covariantly constant with respect to the Levi-Civita connection, while it can be constant with respect to a connection with torsion in the general case. The latter condition allows for a wider class of internal spaces, such as nearly-Kähler and half-flat manifolds. The Heterotic String theory has been recently studied in this general context in refs.[4] and [5]. Six-dimensional nearly-Kähler manifolds are all the non-symmetric six-dimensional coset spaces plus the group manifold  $SU(2) \times SU(2)$  and they have been identified as supersymmetric solutions in the case of type II theories (see e.g. [6]-[10]). In the studies of compactification of the Heterotic Supergravity the use of non-symmetric coset spaces was introduced in [11], and recently developed further in [12]-[15]. Particularly, in [15] it was shown that supersymmetric compactifications of the Heterotic String theory of the form  $AdS_4 \times S/R$  exist when background fluxes and general condensates are present. In addition, effective theories have been constructed in [10], [16], [17] in the case of type II supergravity.

Here we would like to discuss the dimensional reduction of the Heterotic String at leading order in  $\alpha'$  in the case where the internal manifold admits a nearly-Kähler structure. In

section 2 we provide a brief reminder of the Heterotic Supergravity and discuss the basics of manifolds with  $SU(3)$ -structure. In section 3 we present the general reduction procedure that we follow and determine the resulting four-dimensional Lagrangian. In section 4 we apply the previously found results in the case of six-dimensional non-symmetric coset spaces (i.e. in all nearly-Kähler manifolds, omitting the case of the group manifold since it cannot lead to chiral fermions in four dimensions) and we discuss the supergravity description from the four-dimensional point of view. Section 5 contains a discussion on the inclusion of gauge fields in our framework. Finally, our conclusions appear in section 6.

## 2 General Framework

### 2.1 Heterotic Supergravity

In this section we briefly review the field content and the Lagrangian of the Heterotic Supergravity in order to fix our notation and conventions.

The field content of the Heterotic Supergravity consists of the  $\mathcal{N} = 1$ ,  $D = 10$  supergravity multiplet, which accommodates the fields  $g_{MN}$ ,  $\psi_M$ ,  $B_{MN}$ ,  $\lambda$  and  $\varphi$  (i.e. the graviton, the gravitino which is a Rarita-Schwinger field, the two-form potential, the dilatino which is a Majorana-Weyl spinor, and the dilaton which is a scalar). Capital Latin letters denote here ten-dimensional indices.

The corresponding Lagrangian of the ten-dimensional  $\mathcal{N} = 1$  Heterotic Supergravity in the Einstein frame can be written as [18]

$$\mathcal{L} = \mathcal{L}_b + \mathcal{L}_f + \mathcal{L}_{int}, \quad (2.1)$$

where the different sectors of the theory, ignoring the gauge fields and the gaugini at lowest order, are <sup>1</sup>

$$\hat{e}^{-1}\mathcal{L}_b = -\frac{1}{2\hat{\kappa}^2}(\hat{R}\hat{*}\mathbf{1} + \frac{1}{2}e^{-\hat{\phi}}\hat{H}_{(3)} \wedge \hat{*}\hat{H}_{(3)} + \frac{1}{2}d\hat{\phi} \wedge \hat{*}d\hat{\phi}), \quad (2.2)$$

$$\hat{e}^{-1}\mathcal{L}_f = -\frac{1}{2}\hat{\psi}_M\hat{\Gamma}^{MNP}D_N\hat{\psi}_P - \frac{1}{2}\hat{\lambda}\hat{\Gamma}^M D_M\hat{\lambda}, \quad (2.3)$$

$$\begin{aligned} \hat{e}^{-1}\mathcal{L}_{int} = & -\frac{1}{2}\hat{\psi}_M\hat{\Gamma}^N\hat{\Gamma}^M\hat{\lambda}\partial_N\hat{\phi} + e^{-\hat{\phi}/2}\hat{H}_{PQR}\left(\hat{\psi}_M\hat{\Gamma}^{MPQRN}\hat{\psi}_N + 6\hat{\psi}^P\hat{\Gamma}^Q\hat{\psi}^R \right. \\ & \left. - \sqrt{2}\bar{\hat{\psi}}_M\hat{\Gamma}^{PQR}\hat{\Gamma}^M\hat{\lambda}\right) + \text{four-fermion terms}, \end{aligned} \quad (2.4)$$

where we have placed hats in all the ten-dimensional fields to distinguish them from their four-dimensional counterparts which will appear after the reduction. The three-form  $\hat{H}_{(3)}$  is

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<sup>1</sup>Here we use differential form notation for the kinetic terms of the bosons, which will prove to be useful in the course of the reduction.

the field strength for the  $B$ -field, namely  $\hat{H} = d\hat{B}$ . The gamma matrices are the generators of the ten-dimensional Clifford algebra, hence we place hats on them too, while those with more than one indices denote antisymmetric products of  $\Gamma$ s. Also,  $\hat{\kappa}$  is the gravitational coupling constant in ten dimensions, with dimensions (mass)<sup>4</sup>;  $\hat{e}$  is the determinant of the metric and  $\hat{*}$  is the Hodge star operator in ten dimensions. Let us mention that the gravitational constant is defined as  $\hat{\kappa}^2 = 8\pi G_N$ , where  $G_N$  is the Newton constant. As such its relation to the Planck mass is  $\hat{\kappa} = \frac{1}{m_{Pl}}$ , since the Planck mass is  $m_{Pl} = \frac{1}{\sqrt{8\pi G_N}}$ .

## 2.2 Manifolds with $SU(3)$ -structure

CY manifolds were proposed as internal spaces for compactifications in view of the requirement that a four-dimensional  $\mathcal{N} = 1$  supersymmetry is preserved. Namely they admit a single globally defined spinor, which is covariantly constant with respect to the (torsionless) Levi-Civita connection. However, there exists a larger class of manifolds for which the spinor is covariantly constant with respect to a connection with torsion. These are called manifolds with  $SU(3)$ -structure and clearly CY manifolds are a subclass in the category of  $SU(3)$ -structure manifolds.

In particular, in order to define a nowhere-vanishing spinor on a six-dimensional manifold one has to reduce the structure group  $SO(6)$ . The simplest procedure one can follow is to reduce this group to  $SU(3)$ , since then the decomposition of the spinor of  $SO(6)$  reads  $\mathbf{4} = \mathbf{3} + \mathbf{1}$  and the spinor we are looking for is the singlet, let us call it  $\eta$ . Then, we can use  $\eta$  to define the  $SU(3)$ -structure forms, which are a real two-form  $J$  and a complex three-form  $\Omega$  defined as

$$\begin{aligned} J_{mn} &= \mp i \eta_{\pm}^{\dagger} \gamma_{mn} \eta_{\pm}, \\ \Omega_{mnp} &= \eta_{-}^{\dagger} \gamma_{mnp} \eta_{+}, \\ \Omega_{mnp}^{*} &= -\eta_{+}^{\dagger} \gamma_{mnp} \eta_{-}, \end{aligned} \tag{2.5}$$

where the signs denote the chirality of the spinor and the normalization is  $\eta_{\pm}^{\dagger} \eta_{\pm} = 1$ . These forms are globally-defined and non-vanishing and they are subject to the following compatibility conditions

$$\begin{aligned} J \wedge J \wedge J &= \frac{3}{4} i \Omega \wedge \Omega^{*}, \\ J \wedge \Omega &= 0. \end{aligned} \tag{2.6}$$

Moreover, they are not closed forms but instead they satisfy

$$\begin{aligned} dJ &= \frac{3}{4} i (\mathcal{W}_1 \Omega^{*} - \mathcal{W}_1^{*} \Omega) + \mathcal{W}_4 \wedge J + \mathcal{W}_3, \\ d\Omega &= \mathcal{W}_1 J \wedge J + \mathcal{W}_2 \wedge J + \mathcal{W}_5^{*} \wedge \Omega. \end{aligned} \tag{2.7}$$

The expressions (2.7) define the five intrinsic torsion classes, which are a zero-form  $\mathcal{W}_1$ , a two-form  $\mathcal{W}_2$ , a three-form  $\mathcal{W}_3$  and two one-forms  $\mathcal{W}_4$  and  $\mathcal{W}_5$ . These classes completely characterize the intrinsic torsion of the manifold. Note that the classes  $\mathcal{W}_1$  and  $\mathcal{W}_2$  can be decomposed in real and imaginary parts as  $\mathcal{W}_1 = \mathcal{W}_1^+ + \mathcal{W}_1^-$  and similarly for  $\mathcal{W}_2$ .

The classes  $\mathcal{W}_1$  and  $\mathcal{W}_2$  are vanishing when the manifold is complex and furthermore a Kähler manifold has vanishing  $\mathcal{W}_3$  and  $\mathcal{W}_4$ . A Calabi-Yau manifold has all the torsion classes equal to zero. An interesting class of  $SU(3)$ -structure manifolds are called nearly-Kähler manifolds. In this case all the torsion classes but  $\mathcal{W}_1$  are vanishing. This suggests that the manifold is not Kähler and not even complex. A complete list of other classes of  $SU(3)$ -structure manifolds can be found in [19].

Manifolds with  $SU(3)$ -structure in general and nearly-Kähler manifolds in particular have attracted a lot of interest in flux compactifications over the last years. Here we are interested in six-dimensional nearly-Kähler manifolds, which have been classified in [20]. They are the three non-symmetric six-dimensional coset spaces, namely  $G_2/SU(3)$ ,  $Sp_4/(SU(2) \times U(1))_{non-max}$  and  $SU(3)/U(1) \times U(1)$ , plus the group manifold  $SU(2) \times SU(2)$ . It is therefore interesting to perform an explicit reduction over these spaces<sup>2</sup> and determine the resulting effective actions, a task which we shall perform in the forthcoming sections.

### 3 Reduction procedure

In the present section we focus on the bosonic part of the Heterotic Supergravity Lagrangian and perform its reduction from ten to four dimensions over the coset spaces S/R. Since the Kähler potential and the superpotential of the four-dimensional theory can be obtained from the bosonic part, this procedure will be sufficient to find the supergravity description in four dimensions.

In order to reduce the theory we need ansätze for the bosons, namely for the metric, the dilaton and the  $B$ -field. Starting with the metric, our ansatz reads

$$d\hat{s}_{(10)}^2 = e^{2\alpha\varphi(x)}\eta_{mn}e^me^n + e^{2\beta\varphi(x)}\gamma_{ab}(x)e^ae^b, \quad (3.1)$$

where  $e^{2\alpha\varphi(x)}\eta_{mn}$  is the four-dimensional metric and  $e^{2\beta\varphi(x)}\gamma_{ab}(x)$  is the internal metric, while  $e^m$  are the one-forms of the orthonormal basis in four dimensions and  $e^a$  are the left-invariant one-forms on the coset space. In this ansatz we included exponentials which rescale the metric components. This is always needed in order to obtain an action without any prefactor for the Einstein-Hilbert part. We shall see that we need to specify the values of  $\alpha$  and  $\beta$  in order to fulfil this requirement.

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<sup>2</sup>We shall omit in our discussion the case of the group manifold and treat only the coset spaces in the prospect of obtaining chiral fermions when the gauge sector will be added.

We note that the ansatz (3.1) is dictated by two further requirements. Firstly, the metric is required to be  $S$ -invariant. Secondly, the requirement of consistency of our reduction enforces the vanishing of the Kaluza-Klein (KK) fields and allows only the scalar fluctuations [21]-[23]. In particular, tackling the consistency problem, direct calculations lead to the result that when KK gauge fields take values in the maximal isometry group of the coset space,  $S \times N(R)/R$ <sup>3</sup>, the lower-dimensional theory is, in general, inconsistent with the original one. Full consistency of the effective Lagrangian and field equations with the higher-dimensional theory is guaranteed when the KK gauge fields are  $(N(R)/R)$ -valued [22]. However, when the condition  $\text{rank} S = \text{rank} R$  holds the group  $N(R)/R$  is trivial. This is the case for the spaces we consider and therefore the KK gauge fields vanish. Finally, the part of the internal metric  $\gamma_{ab}(x)$  without the exponential has to be unimodular.

Following the standard procedure (see e.g. [24]) for reducing the Einstein-Hilbert action in the case of a coset space and choosing  $\alpha = -\frac{\sqrt{3}}{4}, \beta = -\frac{\alpha}{3}$ , we find that the corresponding part of the reduced Lagrangian reads

$$\mathcal{L} = -\frac{1}{2\kappa^2} \left( R * \mathbf{1} - P_{ab} \wedge * P_{ab} + \frac{1}{2} d\varphi \wedge * d\varphi \right) - V, \quad (3.2)$$

with the potential  $V$  having the form

$$V = -\frac{1}{8\kappa^2} e^{2(\alpha-\beta)\varphi} (\gamma_{ab} \gamma^{cd} \gamma^{ef} f_{ce}^a f_{df}^b + 2\gamma^{ab} f_{da}^c f_{cb}^d + 4\gamma^{ab} f_{iac} f_b^{ic}) * \mathbf{1}, \quad (3.3)$$

where the index  $i$  runs in  $R$  and  $\kappa = \frac{\hat{\kappa}}{\text{vol}_6}$  is the gravitational coupling constant in four dimensions. In the reduced Lagrangian the fields  $P_{ab}$  are defined as

$$P_{ab} = \frac{1}{2} \left[ (\Phi^{-1})_a^c d\Phi_c^b + (\Phi^{-1})_b^c d\Phi_c^a \right], \quad (3.4)$$

with  $\Phi_b^a$  defined through the relation

$$\gamma_{cd} = \delta_{ab} \Phi_c^a \Phi_d^b. \quad (3.5)$$

As such,  $\Phi$  is a matrix of unit determinant, generically containing scalar fields other than  $\varphi$ , and hence there exists a set  $(\Phi^{-1})_a^b$  of fields satisfying

$$(\Phi^{-1})_a^c (\Phi^{-1})_b^d \gamma_{cd} = \delta_{ab}. \quad (3.6)$$

The corresponding kinetic term in (3.2) will provide the kinetic terms for the extra scalars apart from  $\varphi$ , which are generically needed to parametrize the most general  $S$ -invariant metric and appear through the unimodular metric  $\gamma_{ab}(x)$ .

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<sup>3</sup>Here,  $N(R)$  denotes the normalizer of  $R$  in  $S$ , which is defined as  $N = \{s \in S, \quad sRs^{-1} \subset R\}$ . Note that since  $R$  is normal in  $N(R)$  the quotient  $N(R)/R$  is a group.

As far as the higher-dimensional dilaton is concerned, it is trivially reduced by  $\hat{\phi}(x, y) = \phi(x)$ , since it is already a scalar in ten dimensions. This leads to a kinetic term  $-\frac{1}{4\kappa^2}d\phi \wedge *d\phi$  in the reduced Lagrangian.

Finally, concerning the three-form sector of the theory we expand the  $B$ -field on  $S$ -invariant forms of the coset space; namely our ansatz reads

$$\hat{B} = B(x) + b^i(x)\omega_i(y), \quad (3.7)$$

where the index  $i$  counts the number of  $S$ -invariant two-forms. Then it is straightforward to see that the higher-dimensional three-form  $\hat{H} = d\hat{B}$  can be written in terms of four-dimensional fields as

$$\hat{H} = dB + db^i \wedge \omega_i + b^i d\omega_i. \quad (3.8)$$

Let us note here that unlike the case of CY compactifications, where the expansion forms are harmonic and hence closed, here we expand in forms that are not closed and thus an extra term appears in eq.(3.8). Note in addition that at the order we are working it is straightforward to see that  $d\hat{H} = 0$  and therefore our ansatz (3.7) solves the Bianchi identity as it should.

In order to determine the reduced Lagrangian we need to dualize the expression (3.8) with respect to the ten-dimensional Hodge star operator. Then we find

$$\hat{*}_{10}\hat{H} = e^{-6\alpha\varphi} *_4 dB \wedge vol_6 + e^{-2\alpha\varphi-4\beta\varphi} *_4 db^i \wedge *_6 \omega_i + e^{-6\beta\varphi} b^i vol_4 \wedge *_6 d\omega_i. \quad (3.9)$$

Moreover, the determinant of the metric is  $\hat{e} = e^{2\alpha\varphi}$ .

Using the expressions (3.8) and (3.9) in the corresponding term in the Lagrangian we find that the reduced Lagrangian for this sector becomes

$$\begin{aligned} \mathcal{L} = & -\frac{1}{2\kappa^2}e^{-\phi} \left[ \frac{1}{2}e^{-4\alpha\varphi} d\theta \wedge *d\theta + \frac{m}{2}e^{-4\beta\varphi} db^i \wedge *_6 db^i \right. \\ & \left. + \frac{1}{2}e^{4\alpha\varphi} (n_1(b^i)^2 + n_2\epsilon_{ij}b^ib^j) vol_4 \right] \wedge vol_6, \end{aligned} \quad (3.10)$$

where  $\theta$  is the pseudoscalar obtained by duality transformation on  $dB$ , while  $m$ ,  $n_1$  and  $n_2$  are fixed constants defined by

$$\omega_i \wedge *\omega_j = m\delta_{ij}vol_6, \quad d\omega_i \wedge *d\omega_j = (n_1\delta_{ij} + n_2\epsilon_{ij})vol_6. \quad (3.11)$$

Let us conclude this section by adding some comments concerning the possibility of including a background flux for the ten-dimensional field strength  $\hat{H}$ . Since fluxes can be included as additional sources with indices purely in the internal manifold, we have two three-forms at our disposal,  $\rho_1$  and  $\rho_2$ , as we shall see in the concrete examples of the

following section. Therefore one could in principle include in  $\hat{H}$  a term proportional to either  $\rho_1$  or  $\rho_2$  or both. Note that for the spaces we use it always holds that the structure form  $\Omega$  is proportional to a complex linear combination of these three-forms and particularly to  $\rho_2 + i\rho_1$ . However, we can check that the exterior derivative of any invariant two-form is proportional to  $\rho_2$ . This means that the inclusion of a term proportional to  $\rho_2$  is redundant since it can always be absorbed in the definition of the scalar fields  $b^i$ . This property is intimately connected with the fact that in the nearly-Kähler limit the only non-vanishing torsion class is  $\mathcal{W}_1$ . As we discussed in section 2.2,  $\mathcal{W}_1$  can be split in real and imaginary parts. In our cases the real part is always vanishing and there exists only an imaginary part for this torsion class. Therefore the remaining possibility is to introduce a flux proportional to  $\rho_1$ . However, this is a non-closed form. Then, the addition of such a term would mean that the Bianchi identity would fail to hold, since  $d\hat{H}$  would not vanish anymore. This in turn means that at this level no background flux can be added. The situation certainly could change when gauge fields are taken into account.

As a final remark let us note that in refs.[10] and [16] the suitable basis of expansion forms for nearly-Kähler manifolds has been specified and actually coincides with our basis of  $S$ -invariant forms.

## 4 Examples

In this section we specialize the previous discussion in the case of non-symmetric six-dimensional coset spaces, namely  $G_2/SU(3)$ ,  $Sp_4/(SU(2) \times U(1))_{non-max}$  and  $SU(3)/U(1) \times U(1)$ . We determine the potential in four dimensions and we find the corresponding supergravity description by defining the appropriate Kähler potential and superpotential. All these spaces admit a nearly-Kähler structure and therefore our results are to be compared to the results of refs.[4] and [5]. Indeed, as we shall see, our models are realizations of the formalism of the articles [4] and [5]. In particular, the superpotentials we find can be retrieved through the Heterotic Gukov formula found in [4] (see also [25]).

### Geometry and $SU(3)$ -structure

**$G_2/SU(3)$ :** According to ref.[26] this manifold has one  $G_2$ -invariant two-form given by

$$\omega_1 = e^{12} - e^{34} - e^{56}, \quad (4.1)$$



and two  $G_2$ -invariant three-forms expressed as

$$\begin{aligned}\rho_1 &= e^{136} + e^{145} - e^{235} + e^{246}, \\ \rho_2 &= e^{135} - e^{146} + e^{236} + e^{245},\end{aligned}\tag{4.2}$$

in terms of the coset indices  $1 \dots 6$  which correspond to the complement of  $SU(3)$  in  $G_2$ . On the other hand invariant one-forms do not exist. The invariant forms of the coset space are intimately connected to its  $SU(3)$ -structure forms  $J$  and  $\Omega$ . Indeed  $J$  and  $\Omega$  are given by

$$\begin{aligned}J &= R\omega_1, \\ \Omega &= \sqrt{R^3}(\rho_2 + i\rho_1),\end{aligned}\tag{4.3}$$

where  $R$  is the radius of the space and we can immediately deduce that

$$dJ = -\sqrt{3}R\rho_2 = -\sqrt{3}Re(\Omega).\tag{4.4}$$

Then, from the first equation in (2.7) we can read that the torsion classes  $\mathcal{W}_3$  and  $\mathcal{W}_4$  are vanishing as expected, while it is straightforward to see that

$$\mathcal{W}_1 = -\frac{2i}{\sqrt{3}R}.\tag{4.5}$$

Finally, determining that  $d\Omega = \frac{8i}{\sqrt{3}}R(e^{1234} + e^{1256} - e^{3456})$  we can see that the second equation in (2.7) is consistently satisfied with  $\mathcal{W}_1$  as above and  $\mathcal{W}_2 = \mathcal{W}_5 = 0$ . Thus we find that for this coset space the only non-vanishing torsion class is  $\mathcal{W}_1$ , which means that it naturally admits a nearly-Kähler structure without any further conditions.

**$Sp_4/(SU(2) \times U(1))_{non-max}$ :** Here there exist two  $Sp_4$ -invariant two-forms given by

$$\begin{aligned}\omega_1 &= e^{12} + e^{56}, \\ \omega_2 &= e^{34}\end{aligned}\tag{4.6}$$

and two three-forms expressed as

$$\begin{aligned}\rho_1 &= e^{136} - e^{145} + e^{235} + e^{246}, \\ \rho_2 &= e^{135} + e^{146} - e^{236} + e^{245}.\end{aligned}\tag{4.7}$$

The indices  $1 \dots 6$  are coset indices corresponding to the complement of  $SU(2) \times U(1)$  in  $Sp_4$ . As in the previous case invariant one-forms do not exist. The structure forms are given by

$$\begin{aligned}J &= -R_1\omega_1 + R_2\omega_2, \\ \Omega &= \sqrt{R_1^2 R_2}(\rho_2 + i\rho_1),\end{aligned}\tag{4.8}$$

with  $R_1$  and  $R_2$  the radii of the space. The non-vanishing torsion classes in this case are

$$\mathcal{W}_1 = -\frac{2i}{3} \frac{2R_1 + R_2}{\sqrt{R_1^2 R_2}}, \quad (4.9)$$

$$\mathcal{W}_2 = -\frac{4i}{3} \frac{1}{\sqrt{R_1^2 R_2}} [R_1(R_1 - R_2)e^{12} - 2R_2(R_2 - R_1)e^{34} + R_1(R_1 - R_2)e^{56}] \quad (4.10)$$

and it is obvious that the space has a nearly-Kähler limit when the condition  $R_1 = R_2$  is satisfied.

**$SU(3)/U(1) \times U(1)$ :** The coset space  $SU(3)/U(1) \times U(1)$  has three  $SU(3)$ -invariant two-forms given by

$$\begin{aligned} \omega_1 &= e^{12}, \\ \omega_2 &= e^{45}, \\ \omega_3 &= e^{67}, \end{aligned} \quad (4.11)$$

two invariant three-forms expressed as

$$\begin{aligned} \rho_1 &= e^{147} - e^{156} + e^{246} + e^{257}, \\ \rho_2 &= e^{146} + e^{157} - e^{247} + e^{256}, \end{aligned} \quad (4.12)$$

while invariant one-forms do not exist. In this case the indices 3 and 8 correspond to the two  $U(1)$ s and the rest are coset indices corresponding to the complement of  $U(1) \times U(1)$  in  $SU(3)$ . The forms which specify the  $SU(3)$ -structure are

$$\begin{aligned} J &= -R_1\omega_1 + R_2\omega_2 - R_3\omega_3, \\ \Omega &= \sqrt{R_1 R_2 R_3}(\rho_2 + i\rho_1), \end{aligned} \quad (4.13)$$

where the three radii of the space are involved, while the torsion classes are

$$\mathcal{W}_1 = -\frac{2i}{3} \frac{R_1 + R_2 + R_3}{\sqrt{R_1 R_2 R_3}}, \quad (4.14)$$

$$\begin{aligned} \mathcal{W}_2 &= -\frac{4i}{3} \frac{1}{\sqrt{R_1 R_2 R_3}} [R_1(2R_1 - R_2 - R_3)e^{12} - R_2(2R_2 - R_1 - R_3)e^{34} \\ &\quad + R_3(2R_3 - R_1 - R_2)e^{56}]. \end{aligned} \quad (4.15)$$

Again it is straightforward to see that under the condition of equal radii this space admits a nearly-Kähler structure.

## Supergravity description in four dimensions

**$G_2/SU(3)$ :** For  $G_2/SU(3)$  the most general  $G_2$ -invariant metric is given by

$$g_{ab} = e^{2\beta\varphi} \delta_{ab}, \quad (4.16)$$

namely there is only one scale and one scalar field  $\varphi$  parametrizing the internal metric. Thus, using the fact that  $\gamma_{ab} = \delta_{ab}$  as well as the structure constants of this coset space [11], we easily find that the four dimensional potential in this case is

$$V = -\frac{1}{2\kappa^2} (10e^{\frac{8\alpha}{3}\varphi} - 6e^{-\phi+4\alpha\varphi} b^2). \quad (4.17)$$

Note that we found  $n_1 = 12$  for the coefficient  $n_1$  appearing in eq.(3.10) in the present case. In order to bring the reduced Lagrangian in the standard four-dimensional supergravity form we define the complex superfields, consisting of all the scalar moduli,

$$\begin{aligned} S &= e^{\phi_0} + i\lambda, \\ T &= e^{-\varphi_0/\sqrt{3}} + ib. \end{aligned} \quad (4.18)$$

Here for convenience we have redefined two of the moduli as

$$\begin{aligned} \phi_0 &= \frac{1}{2}(-\phi - 4\alpha\varphi), \\ \varphi_0 &= \frac{1}{2}(-\varphi + 4\alpha\phi). \end{aligned} \quad (4.19)$$

Then we claim that the Kähler potential and the superpotential have the form

$$K = \frac{1}{\kappa^2} \left[ -\ln(S + \bar{S}) - 3\ln(T + \bar{T}) \right], \quad (4.20)$$

$$W = 4\sqrt{3}T. \quad (4.21)$$

Indeed we can easily verify that the four-dimensional potential (4.17) results from the supergravity expression

$$V(\Phi, \bar{\Phi}) = \frac{1}{\kappa^4} e^{\kappa^2 K} \left( K^{i\bar{j}} \frac{DW}{D\Phi^i} \frac{D\bar{W}}{D\bar{\Phi}^{\bar{j}}} - 3\kappa^2 W \bar{W} \right), \quad (4.22)$$

where the complex superfields are collectively denoted by  $\Phi$  and the derivatives involved are the Kähler covariant derivatives

$$\frac{DW}{D\Phi^i} = \frac{\partial W}{\partial \Phi^i} + \frac{\partial K}{\partial \Phi^i} W, \quad (4.23)$$

for  $K$  and  $W$  given in eqs.(4.20) and (4.21) respectively. Also one can check that the kinetic terms are exactly retrieved as

$$-\frac{1}{\kappa^2}K_{i\bar{j}}d\Phi^i \wedge *d\bar{\Phi}^{\bar{j}} \quad (4.24)$$

with the same Kähler potential, as required by supergravity.

**$Sp_4/(SU(2) \times U(1))_{non-max}$ :** This coset space admits two independent scales therefore we need to parametrize the metric by an extra scalar field, say  $\chi$ . Then the metric can be written as

$$g_{ab} = e^{2\beta\varphi} \text{diag}(e^{2\gamma\chi}, e^{2\gamma\chi}, e^{-4\gamma\chi}, e^{-4\gamma\chi}, e^{2\gamma\chi}, e^{2\gamma\chi}). \quad (4.25)$$

To ensure the correct kinetic term for the new scalar field we choose  $\gamma^2 = \frac{1}{24}$ . Also, there are now two scalars from the  $B$ -field,  $b_1$  and  $b_2$ . In the same spirit as in the previous case we find that the four-dimensional supergravity description is obtained by defining the Kähler potential and superpotential as follows

$$K = \frac{1}{\kappa^2} \left[ -\ln(S + \bar{S}) - \ln[(T_1 + \bar{T}_1)^2(T_2 + \bar{T}_2)] \right], \quad (4.26)$$

$$W = -2T_1 + T_2, \quad (4.27)$$

where the complex superfields are defined as

$$\begin{aligned} S &= e^{\phi_0} + i\lambda, \\ T_1 &= e^{-\varphi_0/\sqrt{2}} + ib_1, \\ T_2 &= e^{-\chi_0} + ib_2. \end{aligned} \quad (4.28)$$

Note that the redefinitions

$$\phi_0 = -\frac{1}{2}(\phi + 4\alpha\varphi), \quad (4.29)$$

$$\varphi_0 = -\frac{\sqrt{2}}{2}(\phi - 4\alpha\varphi + 4\gamma\chi), \quad (4.30)$$

$$\chi_0 = -\frac{1}{2}(\phi - \frac{2\alpha}{3}\varphi - 4\gamma\chi) \quad (4.31)$$

are needed in order to ensure the consistency of the previous expressions. The nearly-Kähler limit corresponds to the case  $T_1 = T_2$ .

**$SU(3)/U(1) \times U(1)$ :** Here there exist in principle three independent scales, namely there exist three scalars parametrizing the metric fluctuations  $\varphi, \chi, \psi$  and on the other hand there exist three fields  $b_1, b_2, b_3$  in this case. We write for the metric

$$g_{ab} = e^{2\beta\varphi} \text{diag}(e^{2(\gamma\chi+\delta\psi)}, e^{2(\gamma\chi+\delta\psi)}, e^{2(\gamma\chi-\delta\psi)}, e^{2(\gamma\chi-\delta\psi)}, e^{-4\gamma\chi}, e^{-4\gamma\chi}). \quad (4.32)$$

Correctly normalized kinetic terms for  $\chi$  and  $\psi$  are obtained with the choices  $\gamma^2 = \frac{1}{24}$  and  $\delta^2 = \frac{1}{8}$ .

The same logic as in the previous cases leads in the present one to the Kähler potential

$$K = \frac{1}{\kappa^2} \left[ -\ln(S + \bar{S}) - \ln[(T_1 + \bar{T}_1)(T_2 + \bar{T}_2)(T_3 + \bar{T}_3)] \right] \quad (4.33)$$

and the superpotential

$$W = -T_1 + T_2 - T_3. \quad (4.34)$$

The scalar superfields are now defined as

$$S = e^{\phi_0} + i\lambda, \quad (4.35)$$

$$T_1 = e^{-\varphi_0} + ib_1, \quad (4.36)$$

$$T_2 = e^{-\chi_0} + ib_2, \quad (4.37)$$

$$T_3 = e^{-\psi_0} + ib_3$$

with the redefinitions

$$\phi_0 = -\frac{1}{2}(\phi + 4\alpha\varphi) \quad (4.38)$$

$$\varphi_0 = -\frac{1}{2}\left(\phi - \frac{4\alpha}{3}\varphi + 4\gamma\chi + 4\delta\psi\right) \quad (4.39)$$

$$\chi_0 = -\frac{1}{2}\left(\phi - \frac{4\alpha}{3}\varphi + 4\gamma\chi - 4\delta\psi\right) \quad (4.40)$$

$$\psi_0 = -\frac{1}{2}\left(\phi - \frac{4\alpha}{3}\varphi - 8\gamma\chi\right). \quad (4.41)$$

The nearly-Kähler limit is obtained again when  $T_1 = T_2 = T_3$ .

### Vacua

In the preceding examples we found that the reduction from ten dimensions to four at leading order in  $\alpha'$  leads to  $\mathcal{N} = 1$  supergravities in four dimensions. Let us now study the possible vacua of the four-dimensional theory.

Requiring existence of a supersymmetric vacuum, the F-equations

$$\frac{DW}{D\Phi^i} = 0 \quad (4.42)$$

have to be satisfied. Then from eq.(4.22) we deduce that the vacuum energy in four dimensions is negative semidefinite. Thus as long as supersymmetry remains unbroken it is

impossible to find a de Sitter vacuum [27]. Moreover, the possibility to have Minkowski vacuum suggests that the potential in eq.(4.22) vanishes in the vacuum, which in turn means that in addition the condition

$$W = 0 \tag{4.43}$$

has to be satisfied. Unfortunately, the set of equations (4.42) and (4.43) cannot be satisfied in general. For example in the case of  $G_2/SU(3)$  the above requirements lead to the equations

$$W = 0, \tag{4.44}$$

$$D_S W = -\frac{W}{S + \bar{S}} = 0, \tag{4.45}$$

$$D_T W = 4\sqrt{3} - \frac{3W}{T + \bar{T}} = 0, \tag{4.46}$$

which are obviously inconsistent. Therefore either (i) supersymmetry is preserved and the four-dimensional space is not Minkowski or (ii) supersymmetry is broken and the four-dimensional space can be Minkowski. The same result holds in the other two cases. Since we deal with a theory of gravity it is natural to impose that the cosmological constant vanishes, at least at tree level, and elaborate further the option (ii) above. In the case of  $G_2/SU(3)$  inspecting eq.(4.17) we observe that we can tune the potential to vanish by imposing appropriate relations among the vacuum expectation values of the four-dimensional scalar fields. In particular, if we impose

$$5e^{\frac{8\alpha}{3}\langle\varphi\rangle} = 3e^{-\langle\phi\rangle+4\alpha\langle\varphi\rangle} < b >^2 \tag{4.47}$$

or equivalently, in terms of the redefined fields,

$$5e^{-\frac{2}{\sqrt{3}}\langle\varphi_0\rangle} = 3 < b >^2, \tag{4.48}$$

it is straightforward to see that in the vacuum the potential vanishes. Clearly this vacuum is not supersymmetric. Indeed, supersymmetry is spontaneously broken (due to the super-Higgs effect) and the gravitino obtains a mass

$$m_{3/2} = e^{-\langle G \rangle/2}, \tag{4.49}$$

where the function  $G$  is defined as

$$G = K + \ln(|W|^2). \tag{4.50}$$

The graviton of course remains massless and therefore appears a splitting in the supergravity multiplet.

An obvious suggestion in order to go further in the examination of the possible vacua is to take into account the gauge fields, which have been neglected in the present examination, and also to include background fluxes [28]. We comment on that in the following section.

## 5 Inclusion of gauge fields

The results of our analysis indicate the necessity of including gauge fields in our models and thus working at first order in  $\alpha'$ . This is done by coupling the  $\mathcal{N} = 1$  supergravity multiplet to an  $\mathcal{N} = 1$  vector supermultiplet consisting of the gauge fields  $\hat{A}_M$  and their superpartners, the gaugini  $\hat{\chi}$ . It is well-known that the cancelation of anomalies allows only the gauge groups  $E_8 \times E_8$  and  $SO(32)$  [29] and therefore the gauge fields and the gaugini transform in the adjoint representation of one of these gauge groups.

The bosonic part of the ten-dimensional Lagrangian contains the term

$$\hat{e}^{-1} \mathcal{L}_{b,gauge} = -\frac{\alpha'}{2\kappa^2} e^{-\hat{\phi}/2} Tr(\hat{F} \wedge \hat{*}\hat{F}), \quad (5.1)$$

where  $\hat{F}$  is the field strength of the gauge field  $\hat{A}_M$ . Moreover, the three-form  $\hat{H}$  is now given by

$$\hat{H} = d\hat{B} - \alpha'(\hat{\omega}_{YM} - \hat{\omega}_L), \quad (5.2)$$

where the Chern-Simons forms are defined as usual

$$\hat{\omega}_{YM} = Tr(\hat{F} \wedge \hat{A} - \frac{1}{3} \hat{A} \wedge \hat{A} \wedge \hat{A}), \quad (5.3)$$

$$\hat{\omega}_L = Tr(\hat{\theta} \wedge d\hat{\theta} + \frac{2}{3} \hat{\theta} \wedge \hat{\theta} \wedge \hat{\theta}), \quad (5.4)$$

where we denote the spin-connection with  $\hat{\theta}$ . These two corrections are necessary to cancel completely the anomalies (gauge, gravitational and mixed) of  $\mathcal{N} = 1$ ,  $D = 10$  supergravity coupled to Yang-Mills.

In order to dimensionally reduce the full bosonic Lagrangian of the Heterotic String we need an ansatz for the gauge fields. An interesting possibility emerges when one uses the Coset Space Dimensional Reduction (CSDR) scheme [30],[31]. The CSDR is based on the requirement that the gauge fields are not invariant under the isometries of the coset space but their transformation is compensated by a gauge transformation. This requirement restricts the possible ansatze for the gauge fields [21]. Using the CSDR scheme we can benefit from several results that have been accomplished over the years. Among those we refer the possibility to find four-dimensional chiral theories [32], as well as softly broken supersymmetric Lagrangians [33].

Concerning the possibility to obtain realistic models, interesting four-dimensional GUTs have been found in refs [31]-[33] resulting from ten-dimensional  $\mathcal{N} = 1$  supersymmetric  $E_8$  gauge theories using CSDR. Moreover a rather complete classification of the theories obtained starting from the same ten-dimensional theory by CSDR followed by a subsequent application of the Wilson flux breaking mechanism has been recently given in [34]. Obviously,

more possibilities to obtain realistic models might appear when one includes in the study the  $E_8 \times E_8$  as initial gauge group and relaxes the condition that the discrete symmetries are freely-acting, as it was assumed in the above study.

Finally, note that the reduction of the gauge sector of the ten-dimensional theory will lead to an enhanced potential. Then the Kähler potential and the superpotential of the four-dimensional theory will receive  $\alpha'$  corrections. The possible vacua of the extended models have to be explored again. We plan to report on this work in a forthcoming publication.

## 6 Discussion and Conclusions

Here we have explicitly reduced the Heterotic Supergravity from ten dimensions to four at leading order in  $\alpha'$ , i.e. ignoring the gauge fields, using six-dimensional nearly-Kähler manifolds as internal spaces. We have examined three specific models based on the three six-dimensional non-symmetric coset spaces admitting a nearly-Kähler structure and we have determined the resulting four-dimensional effective actions.

From our results we observe that in all three cases the Kähler potential and the superpotential of the resulting four-dimensional supergravities have the same structure and they differ only on the number of scalar moduli appearing in each case. In particular, the volume of the internal space is parametrized by one scalar field for  $G_2/SU(3)$ , while in the cases of  $Sp_4/(SU(2 \times U(1))_{non-max}$  and  $SU(3)/U(1) \times U(1)$  the volume depends on two and three scalar fields respectively. In addition, the scalar fields emerging from the internal components of the  $B$ -field are one, two and three respectively in the three examples we have studied. It is worth noting that the structure of the Kähler potential is exactly the one appearing in the no-scale models of supergravity [35]. No-scale supergravity is an effective theory exhibiting very interesting features such as that it leads to a vanishing cosmological constant at the classical level, dynamical determination of all mass scales in terms of the Planck scale and potentially realistic low-energy phenomenology<sup>4</sup>.

Concerning the vacuum of these models, as long as supersymmetry remains unbroken it is impossible to find a de Sitter vacuum. On the other hand, obtaining a Minkowski vacuum would mean that the cosmological constant in four dimensions vanishes. Examining the conditions which have to be satisfied in the case of unbroken supersymmetry and vanishing vacuum energy we find that there is no such solution in any of our models at this order in  $\alpha'$ . On the other hand, imposing the vanishing of the vacuum energy leads to a non-supersymmetric vacuum with supersymmetry spontaneously broken. In order to enrich our models and look for realistic phenomenology, we are naturally led to work on the

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<sup>4</sup>The no-scale structure as a low-energy limit of superstring theories has been derived before in e.g. [2],[3].



next order in  $\alpha'$  and include gauge fields and non-vanishing fluxes. Then the hope is that working in the context of the CSDR we shall be able to find interesting supergravity GUTs in four dimensions with an appropriate number of fermion generations and soft breaking of supersymmetry. This work is currently in progress.

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## References

- [1] P. Candelas, G. T. Horowitz, A. Strominger and E. Witten, Nucl. Phys. B **258**, 46 (1985).
- [2] E. Witten, Phys. Lett. B **155**, 151 (1985).
- [3] J. P. Derendinger, L. E. Ibanez and H. P. Nilles, Phys. Lett. B **155**, 65 (1985).  
J. P. Derendinger, L. E. Ibanez and H. P. Nilles, Nucl. Phys. B **267**, 365 (1986).
- [4] S. Gurrieri, A. Lukas and A. Micu, JHEP **0712**, 081 (2007) [arXiv:0709.1932 [hep-th]].  
S. Gurrieri, A. Lukas and A. Micu, Phys. Rev. D **70**, 126009 (2004) [arXiv:hep-th/0408121].
- [5] I. Benmachiche, J. Louis and D. Martinez-Pedrera, Class. Quant. Grav. **25** (2008) 135006 [arXiv:0802.0410 [hep-th]].
- [6] D. Lüst and D. Tsimpis, JHEP **0502** (2005) 027 [arXiv:hep-th/0412250].
- [7] A. Tomasiello, JHEP **0506** (2005) 067 [arXiv:hep-th/0502148].
- [8] K. Behrndt and M. Cvetič, Phys. Rev. Lett. **95**, 021601 (2005) [arXiv:hep-th/0403049].
- [9] P. Koerber, D. Lüst and D. Tsimpis, arXiv:0804.0614 [hep-th].
- [10] T. House and E. Palti, Phys. Rev. D **72**, 026004 (2005) [arXiv:hep-th/0505177].

- [11] T. R. Govindarajan, A. S. Joshipura, S. D. Rindani and U. Sarkar, Phys. Rev. Lett. **57**, 2489 (1986). D. Lüüst, Nucl. Phys. B **276**, 220 (1986); T. R. Govindarajan, A. S. Joshipura, S. D. Rindani and U. Sarkar, Int. J. Mod. Phys. A **2**, 797 (1987). L. Castellani and D. Lüüst, Nucl. Phys. B **296**, 143 (1988).
- [12] G. Lopes Cardoso, G. Curio, G. Dall’Agata, D. Lüüst, P. Manousselis and G. Zoupanos, Nucl. Phys. B **652** (2003) 5 [arXiv:hep-th/0211118].
- [13] A. Micu, Phys. Rev. D **70**, 126002 (2004) [arXiv:hep-th/0409008].
- [14] A. R. Frey and M. Lippert, Phys. Rev. D **72**, 126001 (2005) [arXiv:hep-th/0507202].
- [15] P. Manousselis, N. Prezas and G. Zoupanos, Nucl. Phys. B **739**, 85 (2006) [arXiv:hep-th/0511122].
- [16] A. K. Kashani-Poor, JHEP **0711**, 026 (2007) [arXiv:0709.4482 [hep-th]].
- [17] C. Caviezel, P. Koerber, S. Kors, D. Lüüst, D. Tsimpis and M. Zagermann, [arXiv:0806.3458 [hep-th]].
- [18] G. F. Chapline and N. S. Manton, Phys. Lett. B **120**, 105 (1983).
- [19] M. Grana, Phys. Rept. **423**, 91 (2006) [arXiv:hep-th/0509003].
- [20] J.-B. Butruille, arXiv:math.DG/0612655.
- [21] A. Chatzistavrakidis, P. Manousselis, N. Prezas and G. Zoupanos, Phys. Lett. B **656**, 152 (2007) [arXiv:0708.3222 [hep-th]].
- [22] R. Coquereaux and A. Jadczyk, Nucl. Phys. B **276**, 617 (1986).
- [23] A. Chatzistavrakidis, P. Manousselis, N. Prezas and G. Zoupanos, Fortsch. Phys. **56** (2008) 389 [arXiv:0712.2717 [hep-th]].
- [24] M. Cvetič, G. W. Gibbons, H. Lu and C. N. Pope, Class. Quant. Grav. **20** (2003) 5161 [arXiv:hep-th/0306043].
- [25] G. Lopes Cardoso, G. Curio, G. Dall’Agata and D. Lüüst, JHEP **0310**, 004 (2003) [arXiv:hep-th/0306088].
- [26] F. Mueller-Hoissen and R. Stuckl, Class. Quant. Grav. **5**, 27 (1988).
- [27] B. de Wit, D. J. Smit and N. D. Hari Dass, Nucl. Phys. B **283** (1987) 165.

- [28] A. Strominger, Nucl. Phys. B **274** (1986) 253.
- [29] M. B. Green and J. H. Schwarz, Phys. Lett. B **149** (1984) 117.
- [30] P. Forgacs and N. S. Manton, Commun. Math. Phys. **72** (1980) 15. Yu. A. Kubyshin, I. P. Volobuev, J. M. Mourao and G. Rudolph, Leipzig Univ. - KMU-NTZ-89-07 (89,REC.SEP.)
- [31] D. Kapetanakis and G. Zoupanos, Phys. Rept. **219**, 1 (1992).
- [32] G. Chapline and R. Slansky, Nucl. Phys. B **209** (1982) 461.
- [33] P. Manousselis and G. Zoupanos, JHEP **0203** (2002) 002 [arXiv:hep-ph/0111125].
- [34] G. Douzas, T. Grammatikopoulos and G. Zoupanos, [arXiv:0808.3236 [hep-th]].  
G. Douzas, T. Grammatikopoulos, J. Madore and G. Zoupanos, Fortsch. Phys. **56** (2008) 424.
- [35] A. B. Lahanas and D. V. Nanopoulos, Phys. Rept. **145**, 1 (1987). J. R. Ellis, C. Kounnas and D. V. Nanopoulos, Nucl. Phys. B **247**, 373 (1984). E. Cremmer, S. Ferrara, C. Kounnas and D. V. Nanopoulos, Phys. Lett. B **133** (1983) 61.